Models of Nonlinear Growth

PATRICK COULOMBE and JAMES P. SELIG

Abstract

Models for nonlinear growth are not new, but have not been widely applied in the social and behavioral sciences. In this essay, we describe the fundamental issues relevant to choosing and using a nonlinear growth model. We discuss how researchers can go about choosing a model and then focus on the application of two specific nonlinear models: the fractional polynomial model and the piecewise model. We highlight recent work in reparameterization that allows researchers to choose models with parameters tailored specifically to research questions. We also review recent work on the topic of growth rates in nonlinear models that will allow researchers to obtain richer information from the application of nonlinear models. We conclude by pointing out some of the unresolved issues in the use of nonlinear growth models.

MODELS OF NONLINEAR GROWTH

Models of nonlinear growth are used to represent change over time in scores on a variable measured on several occasions. Both linear and nonlinear models can be used to examine change in a variable over time. A key difference between linear and nonlinear change is that in a model for linear change, the rate of change is constant over time (i.e., the slope of the straight line is the same at all occasions); however, in a model for nonlinear change the rate of change can vary across time (Cudeck & Harring, 2007). For example, Figure 1 shows an individual's proportion of successes in a learning task over 10 days. As can be seen, the individual improved her performance dramatically in the first 2 days (reaching a very high proportion of successes), but improved much slower thereafter; in short, improvement over time was not constant.

Throughout this paper, we refer to nonlinear growth models that take the form:

$$y_{ij} = f(\beta_{1i}, \beta_{2i}, \dots, \beta_{ki}, t_{ij}) + \varepsilon_{ij},$$

where subscript i refers to individual i, subscript j the jth occasion of measurement (timepoint), y_{ij} the individual i's observed score at timepoint j, parameters β_{1i} through β_{ki} the regression weights, t_{ij} the time distance to the origin,

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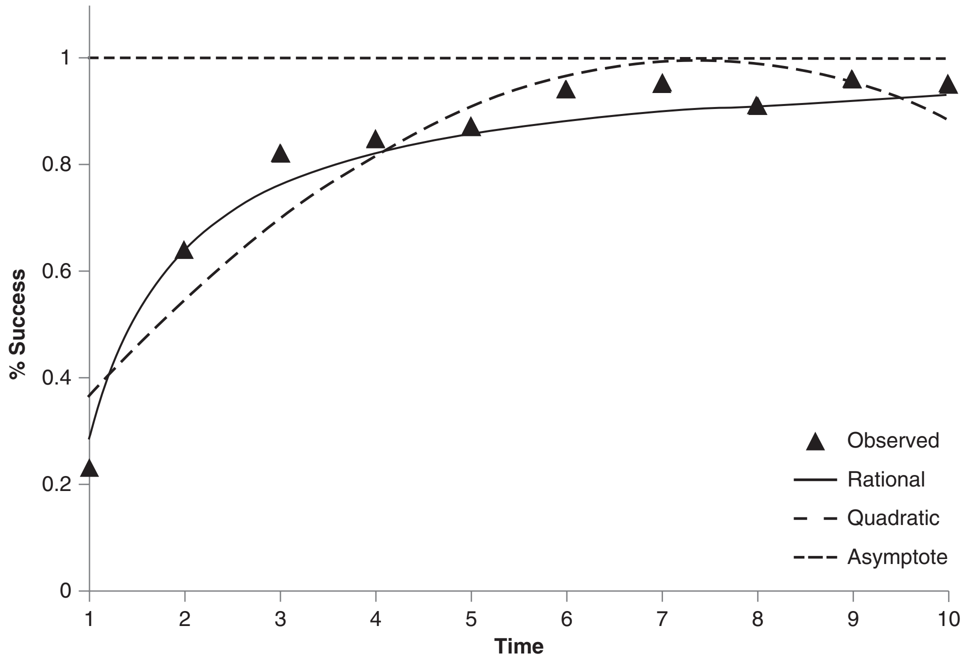

Figure 1 An individual's hypothetical proportion of successes in a learning task over 10 consecutive days, along with a rational and quadratic function fitted to the data. The asymptote is associated with the rational function and has been fixed to 1.

and ε_{ij} the difference between the observed and predicted score for individual i at timepoint j (which is typically assumed to be normally distributed with a mean of 0). The score y_{ij} on the dependent variable is described as a function of time and the regression weights, which may be defined to vary or to be the same across individuals. In a nonlinear growth model, the function f follows a nonlinear trajectory when plotted against time. More formally, if the function is differentiable, its first derivative $f_t'(\beta_{1i}, \beta_{2i}, \ldots, \beta_{ki}, t_{ij})$is itself a function of t_{ij}. Finally, the β parameters can be further specified to be a function of covariates that vary or are fixed over time.

In this essay, we first discuss how a researcher can decide on a particular functional form when modeling nonlinear change over time. We then describe two potentially useful nonlinear growth models, fractional polynomials and piecewise models. Next, we show how a researcher can tailor his or her particular model to substantive questions of interest through reparameterization. Finally, we illustrate how one can effectively use rate of change to give a more complete depiction of the nonlinear growth process to the reader.

CHOOSING A FUNCTIONAL FORM

When modeling linear change over time, the choice of the particular model to use is easy: the linear growth model. When one wants to model nonlinear change over time, however, a vast array of possible models exist. In this section, we discuss how a researcher can choose the functional form of the change over time.

There is no substitute for a researcher's deep understanding of change in a focal variable when choosing an appropriate functional form. This may be especially true when modeling nonlinear change over time, because in many instances, several functions might fit the data equally well. At the very least, the researcher should choose a functional form for the model that is consistent with what he or she knows about change in the variable. For example, an educational researcher examining the cumulative number of aggressive behaviors over the school year among first graders would not choose any function that decreases over time. The researcher's understanding of the variables under study should also drive analytical decisions such as whether to allow a specific regression coefficient to take different values across individuals (Bianconcini, 2012).

Cudeck and Harring (2007) provide three criteria in the selection of an adequate nonlinear function that may support researchers in choosing a functional form: (i) it must fit the data well, (ii) its parameters must have an interpretation that answers interesting substantive questions, and (iii) its characteristic shape corresponds to the hypothesized change (e.g., presence of an upper asymptote if change over time in a variable with a maximum is continually increasing). Only the first one is statistical, while the latter two are grounded in the researcher's understanding of the phenomenon to be modeled. In our view, a common error when choosing functional forms for nonlinear growth models is to overemphasize the first criterion. The point here was that models that fit the data well, yet that do not have interpretable parameters and/or that are not consistent with the researcher's understanding of change, are not useful [even though they fit the data well, is what the point is].

When new variables are examined, or a researcher does not have *a priori* ideas about patterns of change, one very effective way to probe the data is through the use of plots. Cudeck and Harring (2007) mention three different types of plots that can be used in the context of nonlinear growth modeling. The first one is the spaghetti plot, in which the raw scores for a subset of the sample are plotted against time, with straight segments joining two adjacent scores for a given individual. The second one is the swarm plot, in which an estimated function corresponding to the chosen function is plotted against time. Finally, the third one is the trellis plot, which is actually an ensemble of plots arranged in a grid. In each plot, the scores for a single individual are plotted against time, with the estimated curve for that specific individual superimposed (see also Few, 2009, for the description of a similar plotting technique). This quickly gives the researcher (or the reader) an idea of how well the postulated model fits individual data, and of the variability of this fit across individuals.

SPECIFIC CURVE SHAPES

Next, we present two particular variations on nonlinear growth models. The first is the fractional polynomial model and the second is the piecewise model. We chose these two because we believe they are relatively straightforward extensions of models commonly used for growth in the social and behavioral sciences. Although these models are not new, there are relatively few applications of these models.

FRACTIONAL POLYNOMIALS

Researchers interested in modeling nonlinear change over time frequently apply a quadratic curve to the data. In the case of a quadratic curve, the mathematical function fitted to the data is given as

$$y_{ij} = \beta_{0i} + \beta_{1i}t_{ij} + \beta_{2i}t_{ij}^2 + \varepsilon_{ij}.$$

In this model, β_0 represents the predicted score when $t = 0$, β_1 is the rate of change when $t = 0$, and twice the value of β_2 is the magnitude by which the rate of change varies for each one-unit increase in the time variable. An example of a quadratic curve is given in Figure 1 (dashed line), where we attempt to model the individual's increasing proportion of successes in a learning task over time. The quadratic curve follows the observed scores (triangles) quite well. In general, the characteristics of the quadratic curve include presence of a maximum or minimum, symmetry of the curve about the maximum or minimum, and absence of lower or upper asymptotes. These characteristics can be at odds with the researcher's understanding of change. For example, the absence of asymptotes is not consistent with the use of a measure which has a minimum and/or a maximum (in this case, proportion of successes should not exceed 1), and rarely is the pattern of change in behavior symmetric before and after attaining an extreme value.

One way to address these problems is to explicitly select an exponentiation of the time variable that yields a curve consistent with the theory of interest. Polynomial models involving the exponentiation of the time variable with fractions or negative integers are known as *fractional polynomials* (Long & Ryoo, 2010; see also Royston & Altman, 1994; Royston, Ambler, &

Sauerbrei, 1999). The quadratic polynomial shown above is a special case of a fractional polynomial model, in which the time variable is exponentiated to the first and second power.

For example, Figure 1 shows a rational function (solid line) for the individual in attempting to model her increasing proportion of successes over time. The *rational* function takes the general form:

$$y_{ij} = \beta_{0i} + \beta_{1i}t_{ij}^{-1} + \varepsilon_{ij}.$$

Therefore, the rational function is a polynomial function with one-time variable exponentiated to the first negative power. In this model, β_0 represents a horizontal asymptote and β_1 is the negative value of the rate of change when $t = 1$.

As can be seen in Figure 1, this function (with the constraint $\beta_0 = 1$) is consistent with the data, slightly more so than the quadratic curve. Given that we expected performance to improve over time, the quadratic function had the drawback of predicting *decreasing* scores past a certain point (around day 8; see Figure 1). Therefore, we needed a function that would be monotonic, that is, that would increase over time without ever decreasing. The sign of the second parameter of the rational function (β_1) indicates whether scores are increasing ($\beta_1 < 0$; Figure 1) or decreasing ($\beta_1 > 0$) over time. But why did we fix the first parameter (β_0) to be 1 instead of letting it be freely estimated with the data? This parameter in the rational function represents a horizontal asymptote (dashed horizontal line in Figure 1). We needed a function which would allow us to account for the fact that the proportion of successes cannot exceed 1, and the rational function—but not the quadratic function—does just that. We could have allowed for the specific value of the asymptote to be estimated freely. This would allow us to check whether predicted maximal performance differs across individuals, and to potentially predict such individual differences using covariates. For example, we could ask whether maximal performance depends on the age of the person learning the task.

Long and Ryoo (2010) demonstrate graphically the impact of manipulating the exponents of the time variable on the shape of the trajectory over time. These authors separate the models into first-order and second-order polynomials, depending on the number of time variables included in the model. For either type, one can obtain a "flipped" version of any of the curves by reversing the signs of the regression weights (β). And indeed, these curves can be selected to include lower or upper asymptotes (as we did in Figure 1), to be asymmetric about an extreme value, or to depend on scores on covariates (see Lambert, Smith, Jones, & Botha, 2005; Royston & Altman, 1994); overall, fractional polynomials are much more flexible than traditional polynomials in representing a change process.

One difficulty that arises in the use of fractional polynomials over traditional polynomials lies in software use: Fractional polynomials are often more difficult to estimate using standard linear multilevel or structural equation modeling software. Another important issue with regard to the use of fractional polynomial models is the interpretability of the parameters. Oftentimes, the function will contain parameters that do not have straightforward interpretations, and even when the parameters have clear interpretations, the interpretations might not be relevant to the study research questions.

Piecewise Model

The piecewise model (also called spline or multiphase model, sometimes discontinuous model; Singer & Willett, 2003) allows one to model change over time differently from one subset (range) of the time variable to the next subset. There are several ways to achieve this. Most authors have considered piecewise models in which the trajectory for every time subset is exclusively linear. For these models, the nonlinear nature of change is depicted using two or more linear splines. However, it is entirely possible for the individual splines to have nonlinear functional forms.

The simplest piecewise model is a linear–linear model, in which the full range of the period covered by the study is separated into two subsets (phases and epochs), within each of which change over time is linear. What distinguishes the linear trajectory of the two phases is the magnitude of the linear slope. The measurement occasion at which the trajectory changes is called the *knot*, or the *transition point*, and this knot can be fixed or variable across individuals. The function describing the linear–linear model just described is given as

$$y_{ij} = \beta_{0i} + \beta_{1i}t_{1ij} + \beta_{2i}t_{2ij} + \varepsilon_{ij}.$$

In this model, β_0 is the predicted score when both t_1 and $t_2 = 0$, β_1 describes the linear rate of change for the first phase and β_2 is associated with the linear rate of change for the second phase. The novelty in this model is the addition of a *second* time variable. There are two ways to code the two time variables. One way is to treat them as independent: the first time variable starts at a given value and increases until the end of the first phase, at which point it becomes constant for the remainder of measurement occasions, and the second time variable is set to 0 until the last measurement occasion of the first phase, at which point it starts increasing by 1 for each subsequent measurement occasion (Bollen & Curran, 2006). The second way to code the time variables is to make them partially redundant: the first time variable starts at a given value and increases across *all* measurement occasions, and the second variable is coded in the same way as in the first method (Singer & Willett, 2003). For example, for a six-wave study with the first phase spanning times 1 through 3 and the second phase spanning times 4 through 6, the first time variable could be coded (0, 1, 2, 2, 2, 2) in the first method, and (0, 1, 2, 3, 4, 5) in the second method. The second time variable would be coded (0, 0, 0, 1, 2, 3) in both methods.

The interpretation of β_{1i} concerns mainly the first phase, and represents the magnitude of the change in predicted score for each one-unit increase in time during the first phase. However, the same quantity for the second phase depends on how the *first* time variable is coded. In the first method, β_{2i} represents the change in predicted score for each one-unit increase in time during the second phase. In the second method, this quantity is $(\beta_{1i} + \beta_{2i})$ instead.

Piecewise models can also allow for a change in elevation from one phase to the next (Singer & Willett, 2003). An appropriate mathematical function for two phases with a change in elevation is given as

$$y_{ij} = \beta_{0i} + \beta_{1i}t_{1ij} + \beta_{2i}t_{2ij} + \beta_{3i}\ even\ \ t_{ij} + \varepsilon_{ij}.$$

The variable *event* is a binary variable coded as either 0 or 1, which tracks whether a specific event has occurred at measurement occasion *j* for individual *i* (e.g., the individual has graduated, or has started an intervention). Typically, the occurrence of the event will correspond to the change in phases, and will change values only once during the course of the study, although neither condition is a mathematical requirement. The parameter β_{3i} corresponds to the magnitude of the sudden change in predicted score on the dependent variable following the occurrence of the event. In addition, it is possible to model a change in elevation without any change in linear trajectory between the two phases. To do so, one can simply use the second coding method and omit the second time variable from the model (or, equivalently, set β_{2i} to 0).

Extensions to the models presented above are available. For instance, it is possible to model growth over three (or more) distinct phases, which can be particularly useful in a three-phase study involving a baseline, an intervention phase, and a follow-up; the reader is referred to Cudeck and Harring (2007), Ram and Grimm (2007), and Singer and Willett (2003) for examples. One recent interesting extension involved the modeling of phasic change over time that differed across subsets of individuals within a single sample (Kamata, Nese, Patarapichayatham, & Lai, 2012). Another extension is to estimate the timing of the knot with the sample data instead of determining it *a priori* (e.g., Cudeck & Harring, 2007; Harring, Cudeck, & du Toit, 2006). Finally, one possible extension is to allow for nonlinear trajectory *within* each phase. In this case, the function for the piecewise model can be given by the more general formula:

$$y_{ij} = \begin{cases} f_1\left(\beta_{0i}, \ \ldots \ , \beta_{mi}, t\right), & t < \kappa \\ f_2(\beta_{(m+1)i}, \ \ldots \ , \beta_{(m+n)i}, t), & t \geq \kappa \end{cases}.$$

For this general model, scores on the dependent variable are a function of a series of regression coefficients and the time variable, but the function is allowed to change from the first to the second phase at the knot when $t = \kappa$. Therefore, one could model a nonlinear trajectory during the first phase, the second phase, or both the phases, particularly if there are theoretical reasons to do so.

REPARAMETERIZATION

Once a researcher has chosen a specific functional form to use, the goal is to make the model as useful as possible for addressing research questions. This goal can be facilitated through reparameterization. *Reparameterization* is the expression of an existing model using different parameters. The goal of reparameterization is to make the interpretation of the reformulated model more substantively interesting (Preacher & Hancock, in press). The reparameterized model should have the same number of parameters as the original model, and should yield the same fit to the data; in short, the reparameterized model is equivalent to the original model. The choice between two equivalent models should depend on what specific research questions the researcher wishes to address.

Recent literature has offered numerous examples of reparameterized models. Cudeck and du Toit (2002) reexpressed the quadratic model to make the interpretation of its parameters more interesting from a substantive point of view. The resulting function, using our notation, is

$$y_{ij} = \alpha_{1i} - (\alpha_{1i} - \beta_{0i})\left(\frac{t_{ij}}{\alpha_{2i}} - 1\right)^2 + \varepsilon_{ij}.$$

To illustrate the value of reparameterizing a model, let us compare the interpretation of the parameters with those from the classic quadratic model. The two models share one parameter (β_0), which is the predicted score when time is 0. The rest of the parameters differ. In the classic quadratic model, neither β_1 nor β_2 has a straightforward interpretation (except when a researcher is interested in the rate of change at time 0 specifically). Conversely, the reparameterized model bears parameters with interesting interpretations: α_1 is the predicted maximum (or minimum) score (e.g., the maximum proportion of successes), and α_2 is the time at which the predicted maximum (or minimum) is attained (e.g., the required time in days for the maximum performance to be achieved; see Figure 1). Despite these differences, both models fit the data equally well. Therefore, it appears clear from this example that one can substantially improve the usefulness of a model by reparameterizing it so as to make its parameters more readily interpretable.

Although one primary goal of reparameterization is to make parameter estimates more readily interpretable, other advantages can come from reformulating a model. For example, Preacher and Hancock (in press) mention at least three advantages that result from explicitly including in the model a parameter that is of interest to the researcher: (i) estimation will yield a point estimate and a standard error for the parameter estimate, which allows one to conduct hypothesis tests and construct confidence intervals for the parameter; (ii) it allows the parameter of interest to be predicted by other variables included in the model; and (iii) the researcher has the flexibility to treat the newly added parameter as a known or unknown value, and as a value that varies or is fixed across individuals. Another possible advantage is that the reparameterized model can sometimes be estimated more easily.

In addition to the quadratic model, other models have been reparameterized. Choi, Harring, and Hancock (2009) reparameterized the logistic growth model to include parameters such as lower and upper asymptotes, the timepoint when rate of growth is maximal (surge point), and the rate of growth at that timepoint (surge slope). Harring *et al.* (2006) reparameterized a piecewise model so as to explicitly include the knot in the model, making it possible to estimate the location of the knot directly from the sample data. This can be useful if the researcher does not have any hypothesis as to the moment when participants transition from one phase to the next. In the following section, we also mention the recent reparameterization of growth models into growth rate models (Zhang, McArdle, & Nesselroade, 2012).

Many reparameterized models can be challenging to implement in statistical software. Thankfully, some authors provide software syntax necessary to estimate reparameterized models, making these models readily available to researchers (e.g., Choi *et al.*, 2009; Cudeck & du Toit, 2002; Harring *et al.*, 2006; see also Preacher & Hancock, in press). In upcoming years, we expect an increasing number of specific models being reparameterized. For example, Preacher and Hancock (in press) mentioned the possibility of explicitly modeling the angle between the lines of two adjacent phases in a piecewise model. Furthermore, some models can be reformulated into similar, but not fully equivalent models (Biancocini, 2012). In such situations, particularly if differences in model fit between the two models are sizeable, it remains unclear whether a researcher should choose a model based on substantive interpretation of parameters rather than model fit.

RATE OF CHANGE

Since a rate of change that varies over time is precisely what distinguishes linear from nonlinear growth, much is gained from describing rate of change as a function of time. It allows the researcher to gain insights into the growth pattern that he or she is investigating and also provides the reader with a more complete picture of the change over time.

Zhang *et al.* (2012) have reparameterized both the quadratic and exponential classic growth models into *growth rate models*. These models contain parameters, which indicate the rate of change over time. And indeed, they found in their simulation studies that estimation of such models in traditional structural equation modeling software yields accurate parameter estimates, even in the presence of missing data. Since rate of change is now explicitly represented in the models, researchers can ask questions such as "Does rate of change vary across individuals?" and "Is rate of change predicted by covariates?" (e.g., "Is rate of change different for boys and girls?").

When parameter estimates do not carry interesting interpretations, one way to make up for this is to focus on rate of change instead. Such is the approach taken by Long and Ryoo (2010) in describing fractional polynomial models. These authors plot rate of change as a function of time. We present such a plot in Figure 2, corresponding to the rational function shown in Figure 1. From this figure, we can easily see that the individual improves her performance the fastest in the first 2 days of training and with slow improvement thereafter. An even more in-depth presentation of rate of change involves plotting rate of change against time, but with confidence intervals around the curve (e.g., Biesanz, Deeb-Sossa, Papadakis, Bollen, & Curran, 2004; Preacher & Hancock, in press; Zhang *et al.*, 2012). The width of confidence intervals changes as a function of time, and this is reflected in the graphs. This allows the reader to conduct his or her very own hypothesis tests of interest on the rate of change for a series of timepoints, at the level of confidence used in the plots. This also allows the reader to judge the precision of the estimate of rate of change yielded by the reported model as a function of time, all of this in a single visual representation of the growth process. An alternative but easier-to-read version of this plot is to replace the confidence bands with circles around rates of change that are significantly different from 0 at a given significance level (see Preacher & Hancock, in press).

Zhang *et al.* (2012) mention that a task for future research is to reparameterize the nonlinear growth model into a growth acceleration model. *Acceleration of growth* is the rate at which the rate of change varies as a function of time; in mathematical terms, it is represented by the second derivative of the function fitted to the data with respect to time (f_t''), if the derivative function

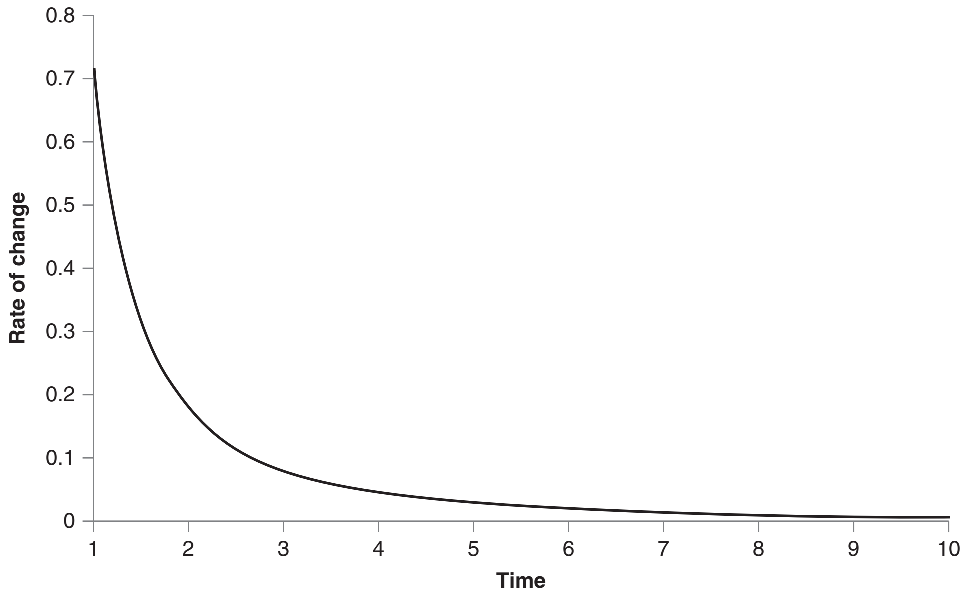

Figure 2 Rate of change in predicted proportion of successes as a function of time.

exists. This acceleration of growth could be a function of time itself, and if this were the case, one could conceivably plot growth acceleration against time, possibly with confidence intervals around the curve. However, acceleration growth models still have to be developed before acceleration of growth becomes part of the standard tools that researchers use to describe a nonlinear growth process.

OTHER RECENT AND FUTURE DEVELOPMENTS

We have addressed several major recent and future developments regarding nonlinear growth modeling in this essay, but this list is not exhaustive. In conclusion, we mention other issues pertaining to nonlinear growth models that we think are worthy of consideration.

One area that needs further work is software development. We have already mentioned in passing that some models require more effort to estimate than others, owing to the nature of their parameters. We can only hope that upcoming versions of software packages will make the transition from simpler to more complex models easier to researchers. Similarly, current software packages cannot allow for the shape of the trajectory to vary across individuals, although this constraint is not a mathematical one (Kamata *et al.,* 2012). One existing problem that we did not mention until now is the presence of discrepancies in results yielded by different software packages. For example, Grimm and Ram (2009) fitted several models using both Mplus and SAS, and found only small differences in estimates of fixed effects, but nonnegligible ones in estimates of random effects. Simulation studies are needed to help determine which software packages yield accurate estimates of random effects. But in the meantime, how is a researcher to choose between two software packages available to them? And indeed, we have no straightforward answer—this decision is currently mainly a matter of personal preference, although one which can affect reported results. Simmons, Nelson, and Simonsohn (2011) have coined such impactful yet personal decisions "researcher degrees of freedom."

Some recent techniques that we have not mentioned until now have the potential to be fruitful in future nonlinear growth modeling endeavors. For instance, Royston and Sauerbrei (2003) used a resampling approach to determine whether predictors should enter a regression model as linear (first-order polynomial) or nonlinear (higher-order polynomial) predictors. To our knowledge, this approach has yet to be applied to growth models. Another novel approach that we delayed mentioning but that *has* been adapted to nonlinear growth models is that of robust growth curve modeling. Zhang, Lai, Lu, and Tong (2012; see also Tong & Zhang, 2012) created an online tool that allows researchers to estimate "robust" growth curve models in the presence of nonnormal data. Their tool permits the estimation of at least one type of nonlinear growth model, namely the latent basis growth curve. Although this free software is still in its infancy, it constitutes a fantastic first step toward dealing with violations of assumptions in nonlinear growth modeling.

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JAMES P. SELIG SHORT BIOGRAPHY

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